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BLOCK-DIAGONALIZATION ANALYSIS OF SYMMETRIC PLATES

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Abstract—This paper presents a block-diagonalization method to solve stiffness equations of isotropic symmetric plates. By means of a suitable "local" coordinate transformation, chosen based on group theory, the stiffness matrix is decomposed into a block-diagonal form. The stiffness equation in the local coordinate is solved block by block, thus realizing numerical efficiency and greatly reducing the requisite amount of computer memory. The efficiency has been further upgraded with the aid of the concept of augmented orbit. This method is applied to a square isotropic plate subject to "asymmetric" loads to show its usefulness.

1. INTRODUCTION

The group-theoretic method has emerged as a systematic means to exploit geometric symmetry. Zloković (1989) has applied it to symmetric structures subjected to asymmetric loads. Bossavit (1986) exploited the symmetry for domains of partial differential equations. Healey (1988) employed it in obtaining equilibrium paths of bifurcation buckling problem of structures with regular-polygonal symmetry; it was also used for large eigenvalue problems for symmetric structures in Healey and Treacy (1991). Dinkevich (1984, 1991) offered a series of complete studies on block diagonalization. Murota and Ikeda (1991) and Ikeda and Murota (1991) used it for bifurcation-tracing analysis of structures with regular-polygonal symmetry, with an emphasis on a dual view point of group symmetry and sparsity.

All these group-theoretic studies are based on a mathematical principle—isotypic decomposition—that the solution space for symmetric structures is to be partitioned into a series of orthogonal subspaces [see, e.g. Serre (1977) and Fujii and Yamaguti (1980)]. This principle guarantees the existence of a transformation matrix H that can put elastic stiffness matrix K of symmetric structures into a block-diagonal form $\tilde{K}$, that is,

$$\tilde{K} \equiv H^{\mathrm{T}} K H = \begin{bmatrix} \tilde{K}_1 & & & 0 \\ & \tilde{K}_2 & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \cdot \end{bmatrix}. \tag{1}$$

Since block-diagonal matrices thus derived are smaller in size compared with the original matrix, this method can save computer memory. At the same time, the cost for sweeping out the matrix is greatly reduced, at the expense of the increase in the cost for matrix multiplication in eqn (1). A procedure to efficiently perform this matrix multiplication compatibly with the matrix finite element method must be developed to put the method fully into practical use.

A typical procedure to reduce this multiplication cost is to exploit the geometric symmetry. For this purpose Bosavit (1986) presented the concept of the "symmetry cell";

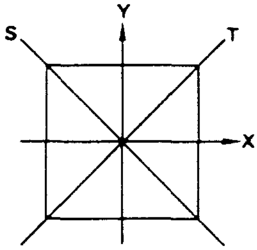

Fig. 1. Geometric symmetry of a square plate.

Dinkevich (1991) employed the "fundamental primitive"; and Murota and Ikeda (1991) used the "orbit" for three-dimensional (truss) structures. The objective of this paper is to augment the techniques to further and fully exploit geometric symmetry. For this purpose, the concept of "augmented" orbit is introduced; the formation of H and the multiplication of eqn (1) are performed systematically to enhance numerical efficiency. The present method is applied to small-displacement analysis of isotropic symmetric plates.

2. THEORY

In this section the group-theoretic method for the block diagonalization of the elastic stiffness matrix of regular-polygonal (n-gonal) plates is presented, as a recapitulation of previous papers, such as Dinkevich (1991) and Murota and Ikeda (1991).

The geometric symmetry of a regular n-gonal plate in the XY-plane can be labeled by the dihedral group†

$$D_n = \{r^k, sr^k \,|\, k = 0, \ldots, n-1\}, \tag{2}$$

of degree n, where the braces $\{\cdot\}$ denote the elements of the relevant group; the element s stands for the reflection with respect to the XZ-plane, and r^k ($k = 0, \ldots, n-1$) for the counter-clockwise rotations around the Z-axis at an angle of $2\pi k/n$ radians; and the multiple denotes that the transformations are performed from the right to the left in sequence.

For a square plate ($n = 4$), for example, r_k ($k = 0, 1, 2, 3$) express the counter-clockwise rotations of $k\pi/2$ radians around the Z-axis; and sr^k ($k = 0, 1, 2, 3$) express the reflections with respect to vertical planes intersecting the XY-plane in the X-, S-, Y- and T-axes, respectively (see Fig. 1).

In the application to the problems of structural engineering, we will extend the notion of group symmetry to the symmetries of (1) material property, (2) plate thickness, (3) finite element meshes‡, and (4) the boundary conditions, in addition to the geometric configuration. In the remainder of this paper, a plate is called D_n-invariant for short if it is D_n-invariant with respect to all those aspects.

The geometric configurations of the deformed state of a D_n-invariant plate can be systematically labeled and categorized by the subgroups of D_n, expressed as:

$$D^j_m = \{r^{kn/m}, sr^{j-1+kn/m} \,|\, k = 0, 1, \ldots, m-1\},$$
$$C_m = \{r^{kn/m} \,|\, k = 0, 1, \ldots, m-1\},$$

where m divides n and $D^1_m = D_m$. Subgroups D^j_m denote reflection symmetric modes with respect to m vertical planes; D^j_m with different j express the same symmetry with the same number of reflection planes but stand for different geometric configurations. C_m are rotation

† In the Schoenflies notation this group is denoted as C_{nv}, whereas D_n means another group in which s represents the half-rotation around the X-axis.

‡ The importance of symmetry in finite-element meshes in describing overall symmetry has been pointed out by Fujii and Yamaguti (1980).

symmetric modes with respect to m rotations of $2k\pi/m$ radians ($k = 0, \ldots, m-1$); C_1 is the asymmetric mode.

Remark

For a D_4-invariant square plate in Fig. 1, D_2 corresponds to a rectangular deformation mode and D_2^2 to a diamond-shaped one. Such a difference is attributable to the difference in the physical meaning of the reflection planes. Likewise D_m^{2k} and D_m^{2k-1} ($k = 1, \ldots, n/2m$) denote physically different modes for n/m even (cf. Ikeda *et al.*, 1991). The symbol D_m^j, accordingly, is used to identify physically different modes, although it is customary to use D_m to represent D_m^j.

Let us denote a system of discretized stiffness equations of a plate by

$$\mathbf{F}(\mathbf{f}, \mathbf{u}) \equiv \mathbf{f} - K\mathbf{u} = \mathbf{0}, \tag{3}$$

where $\mathbf{f}$ is a loading pattern vector; $\mathbf{u}$ denotes a nodal displacement vector; K is an elastic stiffness matrix. The symmetry of $\mathbf{F}$ can be represented by the condition of equivariance with respect to D_n:

$$T(g)\mathbf{F}(\mathbf{f}, \mathbf{u}) = \mathbf{F}(T(g)\mathbf{f}, T(g)\mathbf{u}), \quad \text{for all } g \in D_n, \tag{4}$$

where $T(g)$ is a unitary matrix expressing the transformation associated with an element g of D_n. Then it is easy to show the commutability of K and $T(g)$

$$T(g)K = KT(g), \quad \text{for all } g \in D_n. \tag{5}$$

The condition (4) of equivariance indicates that the transformation $T(g)$ on the independent variables $\mathbf{f}$ and $\mathbf{u}$ will result in rearranging the set of equations $\mathbf{F}$ through the same transformation $T(g)$. Equation (3) for a D_n-invariant plate usually enjoys D_n-equivariance so that one needs not to prove it by showing eqn (4) or (5) [to be precise, the equivariance can be assessed by showing the symmetry law by Dinkevich (1991)].

Symmetric systems have the characteristic that the space of the solution $\mathbf{u}$ of the equilibrium equation (3) is to be decomposed into a series of mutually orthogonal subspaces by so-called isotypic (standard) decomposition [see, e.g. Serre (1977)]. For a D_n-invariant plate, or a D_n-equivariant system (3) to be more precise, this decomposition guarantees the existence of a transformation matrix [see Ikeda and Murota (1991) and Murota and Ikeda (1991)]:

$$H = [H^{s1}, \ldots, H^{sR_s}, H^{d1}, H^{d1-}, \ldots, H^{dR_d}, H^{dR_d-}], \tag{6}$$

which is independent of particular points and puts the stiffness matrix K into a block-diagonal form $\tilde{K}$:

$$\tilde{K} \equiv H^{\mathrm{T}}KH = \mathrm{diag}[\tilde{K}^{s1}, \ldots, \tilde{K}^{sR_s}, \tilde{K}^{d1}, \tilde{K}^{d1}, \ldots, \tilde{K}^{dR_d}, \tilde{K}^{dR_d}]$$

$$= \begin{bmatrix} \tilde{K}^{s1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & \tilde{K}^{dR_d} \end{bmatrix}, \tag{7}$$

where

$$\tilde{K}^{\mu} = (H^{\mu})^{\mathrm{T}} K H^{\mu}, \quad \mu = \mathrm{s}1, \ldots, \mathrm{s}R_{\mathrm{s}}; \mathrm{d}1, \ldots, \mathrm{d}R_{\mathrm{d}}; \tag{8}$$

$$\begin{cases} R_{\mathrm{s}} = 4, \quad R_{\mathrm{d}} = n/2 - 1 & \text{for } n \text{ even}, \\ R_{\mathrm{s}} = 2, \quad R_{\mathrm{d}} = (n-1)/2 & \text{for } n \text{ odd}. \end{cases}$$

Here, $(\cdot)^{\mathrm{T}}$ denotes the transpose†; R_{s} is equal to the number of square diagonal blocks $\tilde{K}^{\mathrm{s}j}$ ($j = 1, \ldots, R_{\mathrm{s}}$) that are not repeated, and R_{d} to the number of $\tilde{K}^{\mathrm{d}j}$ ($j = 1, \ldots, R_{\mathrm{d}}$) that are repeated twice. In group theory R_{s} denotes the number of one-dimensional irreducible representations, and R_{d} that of two-dimensional ones. For brevity we often refer to a submatrix of eqn (6) as H^{μ}, where $\mu = \mathrm{s}j$, $\mathrm{d}j$ or $\mathrm{d}j-$.

Each irreducible representation μ of D_n is defined as

$$\Sigma(\mu) = \{g \in D_n \mid T^{\mu}(g) = I\},$$

where T^{μ} is the irreducible representation matrix for μ. Based on Murota and Ikeda (1991), we have

$$\Sigma(\mathrm{s}1) = D_n; \quad \Sigma(\mathrm{s}2) = C_n; \quad \Sigma(\mathrm{s}3) = D_{n/2}; \quad \Sigma(\mathrm{s}4) = D^{2}_{n/2};$$
$$\Sigma(\mathrm{d}j) = C_{\gcd(j,n)}, \quad j = 1, \ldots, R_{\mathrm{d}}, \tag{9}$$

where $\gcd(j, n)$ is the greatest common divisor of j and n.

We can elaborately choose $H^{\mathrm{d}j}$ to be symmetric with respect to $D^{k}_{\gcd(j,n)}$ irrespective of the parity of $n/\gcd(j,n)$, and $H^{\mathrm{d}j-}$ to $D^{k+n/(2\gcd(j,n))}_{\gcd(j,n)}$ for $n/\gcd(j,n)$ even. Such a choice, combined with the concept of "augmented orbit" in Section 4.2, will achieve numerical efficiency.

Then consider a local coordinate transformation

$$\mathbf{u} = H\mathbf{w} = \sum_{\mu} H^{\mu}\mathbf{w}^{\mu} = \sum_{j=1}^{R_{\mathrm{s}}} H^{\mathrm{s}j}\mathbf{w}^{\mathrm{s}j} + \sum_{j=1}^{R_{\mathrm{d}}} (H^{\mathrm{d}j}\mathbf{w}^{\mathrm{d}j} + H^{\mathrm{d}j-}\mathbf{w}^{\mathrm{d}j-}), \tag{10}$$

with a local coordinate variable

$$\mathbf{w} = [(\mathbf{w}^{\mathrm{s}1})^{\mathrm{T}}, \ldots, (\mathbf{w}^{\mathrm{s}R_{\mathrm{s}}})^{\mathrm{T}}, (\mathbf{w}^{\mathrm{d}1})^{\mathrm{T}}, (\mathbf{w}^{\mathrm{d}1-})^{\mathrm{T}}, \ldots, (\mathbf{w}^{\mathrm{d}R_{\mathrm{d}}})^{\mathrm{T}}, (\mathbf{w}^{\mathrm{d}R_{\mathrm{d}}-})^{\mathrm{T}}]^{\mathrm{T}}.$$

By means of the transformation (10), the stiffness equation (3) can be partitioned into a number of distinct equations for each block

$$(H^{\mathrm{s}j})^{\mathrm{T}}\mathbf{f} = \tilde{K}^{\mathrm{s}j}\mathbf{w}^{\mathrm{s}j}, \quad j = 1, \ldots, R_{\mathrm{s}}; \tag{11}$$

$$(H^{\mathrm{d}j})^{\mathrm{T}}\mathbf{f} = \tilde{K}^{\mathrm{d}j}\mathbf{w}^{\mathrm{d}j}, \quad (H^{\mathrm{d}j-})^{\mathrm{T}}\mathbf{f} = \tilde{K}^{\mathrm{d}j}\mathbf{w}^{\mathrm{d}j-}, \quad j = 1, \ldots, R_{\mathrm{d}}. \tag{12}$$

Substitution of the solution $\mathbf{w}^{\mu}$ of eqns (11) and (12) into eqn (10) leads to the solution $\mathbf{u}$ of the original equation

$$\mathbf{u} = \sum_{\mu} H^{\mu}(\tilde{K}^{\mu})^{-1}(H^{\mu})^{\mathrm{T}}\mathbf{f} \tag{13}$$

as a superposition of the solutions for each block.

† It is suffice to use the transpose $(\cdot)^{\mathrm{T}}$ here, while in a more general context the Hermitian conjugate $(\cdot)^{\mathrm{H}}$ is used.

3. FORMULATION OF TRANSFORMATION MATRIX

In this section we present a systematic method to formulate a transformation matix H for a D_n-invariant plate based on the concept of orbit [see Golubitsky and Schaeffer (1985)].

A set of nodal points for a D_n-invariant plate as a whole is also D_n-invariant. This whole set of nodal points can be partitioned into a series of D_n-invariant subsets of points, which are called "orbits" in mathematical terminology. Elements r^k and sr^k ($k = 0, \ldots, n-1$) of D_n transform a nodal point x to points r^kx and (sr^k)x, respectively, where x expresses its location vector. In this manner one can define an orbit consisting of a set of points:

$$\{r^k\mathrm{x} \quad \text{and} \quad (sr^k)\mathrm{x} \mid k = 0, \ldots, n-1\},$$

which can be transformed to one another and remain unchanged as a whole by the geometric transformation by the elements of D_n. As shown in Fig. 2, the following four types of orbits exist: (1) Center type, (2) n-gon type I, (3) n-gon type II, and (4) $2n$-gon type. In particular, for a square plate ($n = 4$), the n-gon type is called the square one, and the $2n$-gon type the octagonal one. Figure 3 displays an example of orbital decomposition of a meshed square plate, which is made up of one center type, two square type I, two square type II, and one octagonal type. Just as the finite element method expresses a structure as an assemblage of elements, a meshed plate is expressed as an assemblage of orbits.

The basic strategy which we will employ is to formulate each column vector of H orbit by orbit. Such a definition not only makes its formulation simpler but also makes the resulting H matrix sparser, since only the components associated with the orbit are nonzero. The orbital decomposition presented in this paper is numerically more efficient than that by Zloković (1989) in that the former decomposes a plate into a greater number of orbits and makes H sparser.

For $n = 4$, for example, we can choose the deformation modes shown in Fig. 4 for each orbit corresponding to each μ [see Murota and Ikeda (1991) or Ikeda and Murota (1991)]. Here the arrows denote the direction of the deformation vector; the solid lines express deformed states and the dashed lines the initial state. Some of these deformation modes enjoy the antisymmetry with respect to some rotations and reflections. For example, the modes for s2 have no reflection symmetries. However, they are antisymmetric with respect to the four reflections sr^k ($k = 0, 1, 2, 3$) in that they reverse the signs of deformations but retain their magnitudes unchanged. Further the modes for d1 − have been elaborately

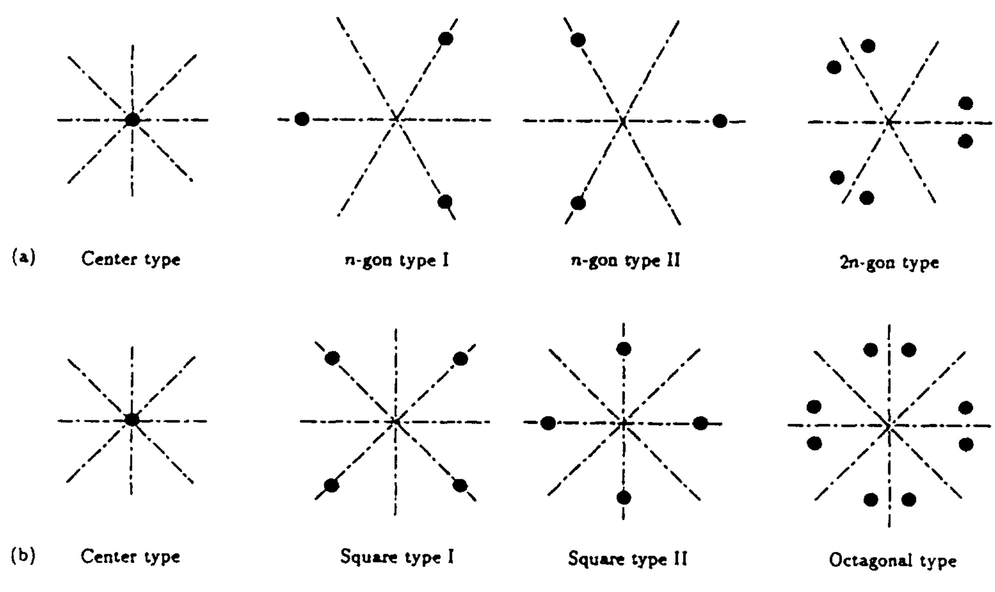

Fig. 2. Orbital decomposition of nodal points. (a) $n = 3$; (b) $n = 4$. Dotted-dash lines: planes of reflection symmetry.

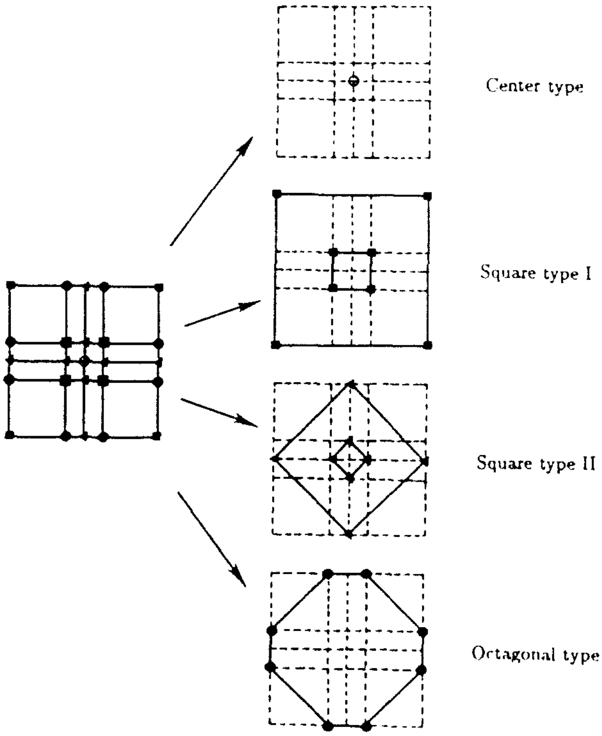

Fig. 3. Example of orbital decomposition of a meshed square plate.

chosen so as to enjoy invariance with respect to sr^2 and also antisymmetry with respect to s and r^2. Such antisymmetry will be exploited in the following section to enhance the numerical efficiency of the present method.

4. EFFICIENT COMPUTATION OF LOCAL STIFFNESS MATRIX

In this method we have greatly reduced the cost for sweeping out by replacing the analysis for K in eqn (3) by that of $\tilde{K}^{\mu}$ of eqn (13) with smaller sizes, at the expense of the

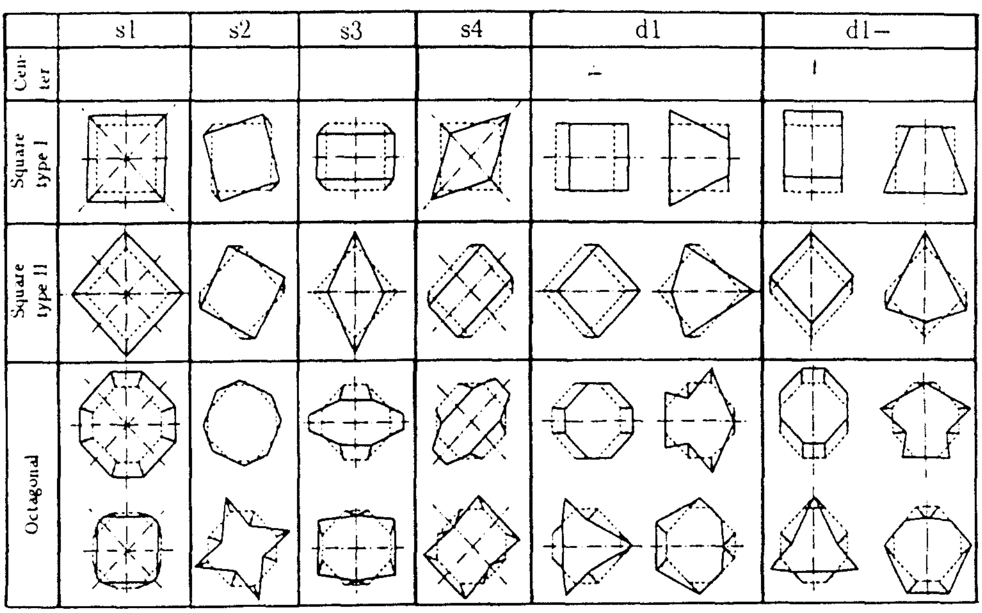

Fig. 4. Deformation modes for each orbit ($n = 4$). Dotted-dash lines: planes of reflection symmetry.

additional cost for the matrix multiplication in eqn (8) to compute $\tilde{K}^\mu$. In order to further enhance its efficiency this additional cost is reduced here through the procedures listed below, details of which will be given in the following subsections:

(1) Because only the components $\mathbf{w}^\mu$ with nonzero $(H^\mu)^T\mathbf{f}$ will contribute to the final solution of eqn (13), only $\tilde{K}^\mu$ with nonzero $(H^\mu)^T\mathbf{f}$ need to be computed and sweeped out.

(2) Equation (8) is used to compute $\tilde{K}^\mu$, instead of eqn (7), so as to reduce the size of matrices and in turn to reduce the multiplication cost.

(3) The matrix multiplication of eqn (8) is to be performed for each element and then to be superposed for the whole plate so as to reduce the size of matrices.

(4) The matrix multiplication for all elements needs not to be performed if the geometric symmetry (D_n-invariance) of the plate is to be implemented.

(5) For twice repeated matrices $\tilde{K}^{d_j}$ in eqn (7), only one of them should be computed and be sweeped out.

(6) Because $\tilde{K}^\mu$ are symmetric matrices, only their upper triangular parts are to be computed.

4.1. *Transformation for each element*

The column of H^μ is associated with the variable $\mathbf{w}^\mu$, and its row with global variable $\mathbf{u}$. Remembering that the components of $\mathbf{w}^\mu$ correspond to the deformation modes of orbits, one can decompose $\mathbf{w}^\mu$ into orbital components:

$$\mathbf{w}^\mu = ((\mathbf{w}_1^\mu)^T, \ldots, (\mathbf{w}_{N_0}^\mu)^T)^T, \tag{14}$$

where $N_0 = N_0(\mu)$ is the number of orbits; and $\mathbf{w}_i^\mu$ is the ith orbit ($i = 1, \ldots, N_0$). Likewise $\mathbf{u}$ is decomposed into nodal components:

$$\mathbf{u} = ((\mathbf{u}_1)^T, \ldots, (\mathbf{u}_{N_P})^T)^T, \tag{15}$$

where N_P is the number of nodal points; and $\mathbf{u}_i$ is the ith point ($i = 1, \ldots, N_P$).

To be consistent with these decomposed variables of eqns (14) and (15), the submatrix H^μ, the block-diagonal stiffness matrices $\tilde{K}^\mu$, and the stiffness matrix K are partitioned into block matrices as below.

$$H^\mu = (H_{ij}^\mu \mid i = 1, \ldots, N_P;\ j = 1, \ldots, N_0)$$

$$= \begin{pmatrix} H_{1,1}^\mu & \cdots & H_{1,N_0}^\mu \\ \vdots & & \vdots \\ H_{N_P,1}^\mu & \cdots & H_{N_P,N_0}^\mu \end{pmatrix},$$

$$\tilde{K}^\mu = (\tilde{K}_{ij}^\mu \mid i, j \ldots, N_0),$$

$$K = (K_{ij} \mid i, j = 1, \ldots, N_P).$$

Let the plate be made up of N_e elements, and the eth element ($e = 1, \ldots, N_e$) consists of M nodal points with node numbers $(p_i \mid i = 1, \ldots, M)$, and denote by

$$(q_i \equiv q_i(p_i) \mid i = 1, \ldots, M)$$

the orbit number of each point. We partition the eth element stiffness matrix, say K^e ($e = 1, \ldots, N_e$), into block matrices as

$$K^e = (K_{p_i p_j}^e \mid i, j = 1, \ldots, M).$$

Then the multiplication of eqn (8) for the whole structure can be replaced by the assemblage of the multiplications for elements

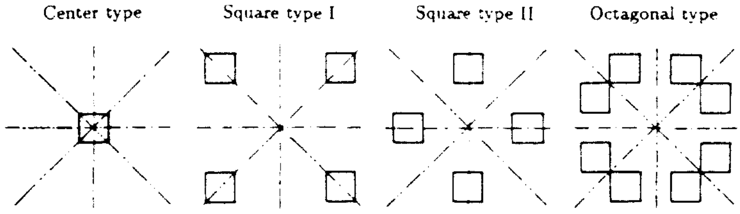

Fig. 5. D_n-invariant orbits among elements ($n = 4$). Dotted-dash lines: planes of reflection symmetry.

$$\tilde{K}^{\mu}_{ij} = \sum_{e=1}^{N_e} \sum_{k=1}^{M} \sum_{l=1}^{M} \tilde{K}^{e\mu}_{q_k q_l} \delta_{iq_k} \delta_{jq_l}, \tag{16}$$

where δ_{iq_k} and δ_{jq_l} denote Kronecker's delta; and

$$\tilde{K}^{e\mu}_{q_k q_l} = (H^{\mu}_{q_k p_k})^T K^{e}_{p_k p_l} H^{\mu}_{q_l p_l}. \tag{17}$$

It is suffice to compute $\tilde{K}^{\mu}_{ij}$ with $i \geqslant j$ in eqn (16), since $\tilde{K}^{\mu}$ is a symmetric matrix. Because the size of matrices in the right-hand side of eqn (17) is much smaller than that in eqn (8), the cost for matrix multiplication has been greatly reduced.

4.2. *Augmented orbit among elements*

The geometric symmetry of a D_n-invariant plate is exploited here by means of the concept of "augmented" orbit among elements. To be consistent with the four types of orbits among "points", the set of elements of such a plate can be decomposed with the use of four types of D_n-invariant orbits among elements shown in Fig. 5 for $n = 4$: (1) center type, (2) n-gon (square) type I, (3) n-gon (square) type II, and (4) $2n$-gon (octagonal) type. It is to be noted that the definition of the orbit varies with the group under which the orbit is invariant. For the subgroups $G \equiv \Sigma(\mu)$ of D_n, the four types of D_n-invariant orbits are decomposed into different kinds of orbits. An example for $G = D_1$ is shown in Fig. 6(a), where the elements ascribed with the same symbol belong to the same orbit.

Say two elements e and e^* belong to the same orbit among "elements" under the action of a subgroup G, and p_k and p_k^* (respectively, p_l and p_l^*) are the points on each of these two elements, respectively, belonging to the same orbit among "points" under the same group action G. Then there exists an orthogonal transformation $T(g)$ for an element g of the subgroup G such that

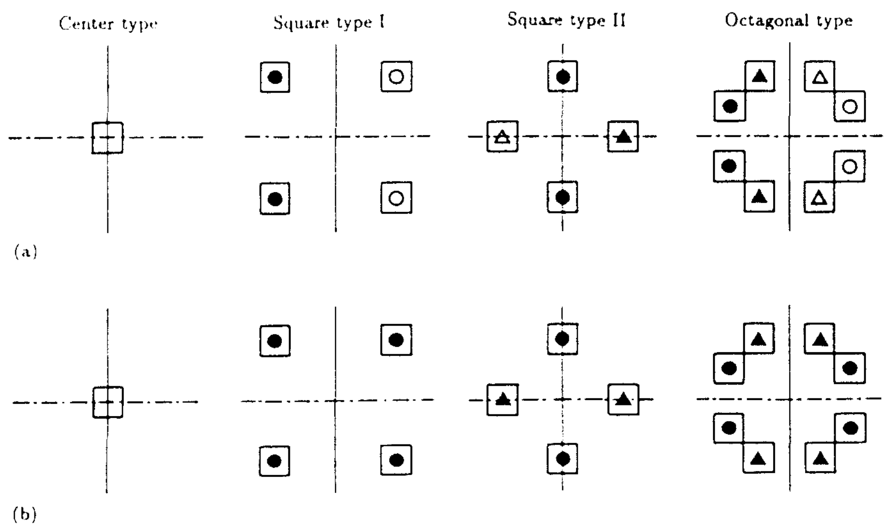

Fig. 6. (a) D_1-invariant orbits among elements; (b) D_1-invariant augmented orbits among elements. Dotted-dash lines: planes of reflection symmetry; solid lines: planes of reflection antisymmetry.

$$T(g)H^{\mu}_{q_\eta p_\eta} = H^{\mu}_{q^*_\eta p^*_\eta}, \quad \eta = k \text{ or } l, \quad g \in G, \tag{18}$$

$$K^{e^*}_{p^*_k p^*_l} = T(g)^{\mathrm{T}} K^{e}_{p_k p_l} T(g), \quad g \in G, \tag{19}$$

where $K^{e^*}_{p^*_k p^*_l}$ is the submatrix for the element stiffness matrix for the element e^* associated with the two points p^*_k and p^*_l.

With the use of eqns (18) and (19), the component $\tilde{K}^{e\mu}_{q_k q_l}$ of eqn (16) can be expressed as:

$$\begin{aligned}\tilde{K}^{e\mu}_{q_k q_l} &\equiv (H^{\mu}_{q_k p_k})^{\mathrm{T}} K^{e}_{p_k p_l} H^{\mu}_{q_l p_l} \\ &= (H^{\mu}_{q_k p_k})^{\mathrm{T}} T(g)^{\mathrm{T}} T(g) K^{e}_{p_k p_l} T(g)^{\mathrm{T}} T(g) H^{\mu}_{q_l p_l} \\ &= [T(g)H^{\mu}_{q_k p_k}]^{\mathrm{T}} [T(g)^{\mathrm{T}} K^{e}_{p_k p_l} T(g)]^{\mathrm{T}} [T(g) H^{\mu}_{q_l p_l}] \\ &= (H^{\mu}_{q^*_k p^*_k})^{\mathrm{T}} K^{e^*}_{p^*_k p^*_l} H^{\mu}_{q^*_l p^*_l} \\ &\equiv \tilde{K}^{e\mu}_{q^*_k q^*_l}\end{aligned} \tag{20}$$

by $[T(g)]^{\mathrm{T}} = [T(g)]^{-1}$. Equation (20) indicates that the elements belonging to the same orbit under the action of G have the same tributary element stiffness for G.

Here, we extend the concept of orbit. Remember that the base vectors of Fig. 4 are selected in such a manner that they often enjoy antisymmetry with respect to the group action, that is,

$$T(g)H^{\mu}_{q_\eta p_\eta} = -H^{\mu}_{q^*_\eta p^*_\eta}, \quad \eta = k \quad \text{or} \quad l, \quad g \in G. \tag{21}$$

Then it is straight forward to show that eqn (20) holds for this case as well. We call a set of points which satisfy either eqn (18) or (21) an "augmented orbit" that is invariant up to the sign. Since the elements belonging to the same augmented orbit have the same tributary stiffness, the stiffness for one representative element among these elements should be evaluated and then multiplied by the number of elements among the augmented orbit to arrive at the stiffness for the whole orbit.

For example, for $n = 4$, Table 1 lists this number for each orbit; and Fig. 6(b) displays the decomposition of the four types of D_4-invariant orbits into D_1-invariant augmented orbits. As can be seen, these augmented orbits consist of a greater number of elements than do customary orbits. The use of the augmented orbits will enhance the numerical efficiency of the present method.

5. EXAMPLES

5.1. *One-element square plate*

As a simple example, we consider a D_4-invariant square plate with a uniform thickness t, and of isotropic material property with constant Young's modulus E and Poisson's ratio ν.

Table 1. Number of elements belonging to augmented orbit among elements for $n = 4$

	μ		
Type of orbit	s1, s2, s3, s4	d1	d1 −
Center	1	1	1
Square type I	4	4	2
Square type II	4	2	4
Octagonal	8	4	4

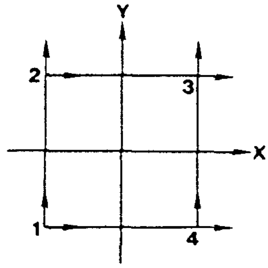

Fig. 7. An isotropic, four-node, square plate element.

We simulate this plate by a four-node finite element with eight degrees of freedom shown in Fig. 7.

By eqn (5) an element stiffness matrix K^e of the D_4-invariant plate must commute with the transformation matrices $T(r^k)$ and $T(sr^k)$ $(k = 0, 1, 2, 3)$ for all elements of D_4. For example, $T(r)$ and $T(s)$ read:

$$T(r) = \begin{bmatrix} & & & & & & & -1 \\ & & & & & & 1 & \\ & -1 & & & & & & \\ 1 & & & & & & & \\ & & & -1 & & & & \\ & & 1 & & & & & \\ & & & & & -1 & & \\ & & & & 1 & & & \end{bmatrix},$$

$$T(s) = \begin{bmatrix} & & 1 & & & & & \\ & & & -1 & & & & \\ 1 & & & & & & & \\ & -1 & & & & & & \\ & & & & & & 1 & \\ & & & & & & & -1 \\ & & & & 1 & & & \\ & & & & & -1 & & \end{bmatrix}.$$

This condition (5) restricts the form of K^e as follows:

$$K^e = \begin{bmatrix} C_1 & & & & & & & \text{Symmetric} \\ C_4 & C_1 & & & & & & \\ C_7 & -C_3 & C_1 & & & & & \\ C_3 & C_5 & -C_4 & C_1 & & & & \\ C_2 & -C_6 & C_5 & -C_3 & C_1 & & & \\ -C_6 & C_2 & C_3 & C_7 & C_4 & C_1 & & \\ C_5 & C_3 & C_2 & C_6 & C_7 & -C_3 & C_1 & \\ -C_3 & C_7 & C_6 & C_2 & C_3 & C_5 & -C_4 & C_1 \end{bmatrix}. \qquad (22)$$

where $(C_i \mid i = 1, \ldots, 7)$ are constants.

For example, an isotropic element stiffness matrix for a Serendipity square element with four nodes is of the form of eqn (22) with

$$C_i = \frac{Et}{12(1-\nu^2)} c_i, \quad i = 1, \ldots, 7,$$

where

$$c_1 = 2(3-\nu), \quad c_2 = -(3-\nu), \quad c_3 = \tfrac{3}{2}(1-3\nu),$$

$$c_4 = c_6 = \tfrac{3}{2}(1+\nu), \quad c_5 = -(3+\nu), \quad c_7 = 2\nu.$$

This element stiffness matrix, therefore, commutes with $T(g)$ and remains invariant under $T(g)$.

The transformation matrix H^{μ} ($\mu = s_1$, s2, s3, s4, d1, or d1−) of eqn (6) for this element, made up of an orbit of square type I, can be given by the deformation patterns of this orbit shown in Fig. 4, that is,

$$H^{s1} = \begin{bmatrix} -\sqrt{2}/4 \\ -\sqrt{2}/4 \\ -\sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{2}/4 \\ -\sqrt{2}/4 \end{bmatrix}, \quad H^{s2} = \begin{bmatrix} \sqrt{2}/4 \\ -\sqrt{2}/4 \\ -\sqrt{2}/4 \\ -\sqrt{2}/4 \\ -\sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{2}/4 \end{bmatrix}, \quad H^{s3} = \begin{bmatrix} -\sqrt{2}/4 \\ \sqrt{2}/4 \\ -\sqrt{2}/4 \\ -\sqrt{2}/4 \\ \sqrt{2}/4 \\ -\sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{2}/4 \end{bmatrix}, \quad H^{s4} = \begin{bmatrix} -\sqrt{2}/4 \\ -\sqrt{2}/4 \\ \sqrt{2}/4 \\ -\sqrt{2}/4 \\ \sqrt{2}/4 \\ \sqrt{2}/4 \\ -\sqrt{2}/4 \\ \sqrt{2}/4 \end{bmatrix},$$

$$H^{d1} = \begin{bmatrix} 1/2 & 0 \\ 0 & -1/2 \\ 1/2 & 0 \\ 0 & 1/2 \\ 1/2 & 0 \\ 0 & -1/2 \\ 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}, \quad H^{d1-} = \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \\ 0 & 1/2 \\ 1/2 & 0 \\ 0 & -1/2 \\ 1/2 & 0 \\ 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

Then K^e can be put into a block-diagonal form with the square blocks of eqn (8):

$$\tilde{K} \equiv H^{\mathrm{T}}KH = \mathrm{diag}\,[\tilde{K}^{s1}, \tilde{K}^{s2}, \tilde{K}^{s3}, \tilde{K}^{s4}, \tilde{K}^{d1}, \tilde{K}^{d1}],$$

with

$$\tilde{K}^{s1} = \frac{Et}{1-\nu}, \quad \tilde{K}^{s2} = 0, \quad \tilde{K}^{s3} = \frac{Et}{1+\nu}, \quad \tilde{K}^{s4} = \frac{Et(2-3\nu)}{2(1-\nu^2)},$$

$$\tilde{K}^{d1} = \frac{Et(3-\nu)}{6(1-\nu^2)} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The three zero diagonals of this matrix are associated with the rigid body displacements of the plate.

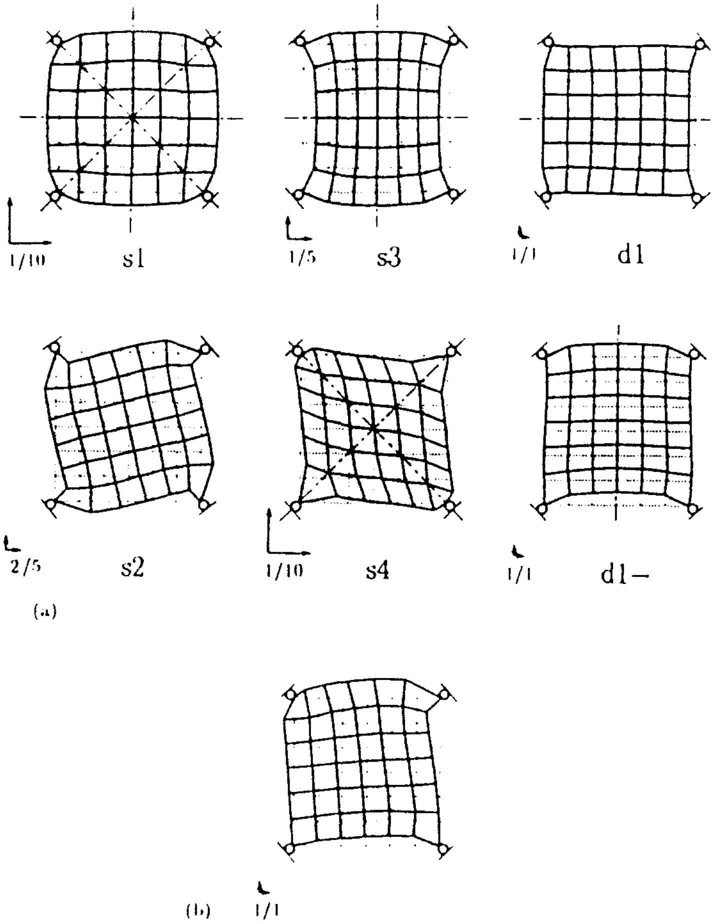

Fig. 8. Deformation modes of a square plate. (a) Deformation in local decomposed coordinates; (b) Deformation in global coordinates.

5.2. *Square plate with N × N meshes*

We consider a D_4-invariant square plate† of a uniform thickness and isotropic material property with $N \times N$ uniform meshes supported at the four corners with point bearing. This plate is subject to an asymmetric load, for which all external force vectors $(H^{\mu})^{T}\mathbf{f}$ are all nonzero. The element stiffness matrix is given by the Serendipity square element with four nodes. The stiffness matrix $\tilde{K}$ transformed by eqn (7) is partitioned into six diagonal blocks.

For $N = 6$ the deformation modes for the solutions $H^{\mu}\mathbf{w}^{\mu}$ of eqn (13) for blocks are shown in Fig. 8(a), where the size of axis denotes the scale for each deformation mode. Then these modes are superposed to arrive at the final solution **u** shown in Fig. 8(b), which is identical with that obtained directly from eqn (3).

In order to demonstrate the numerical efficiency of the present method, we compare it with the direct (conventional) method [similar comparison was done by Dinkevich (1991)]. The revised Cholesky method‡ is used to sweep out stiffness matrices. The cost to sweep out a banded matrix of size M is given by

$$B^2(M-B)+B^3/6 \tag{23}$$

in terms of the number of arithmetic operations, where B is a half band width. This cost for the direct method is evaluated to

† The finite element solution for this plate strongly depends on mesh and would diverge for infinitesimally small mesh widths.

‡ Although the Cholesky method is employed here, it does not necessarily mean that this method is superior to many other alternatives, such as the one-way dissection method by George and Liu (1981) and the block elimination method by Dinkevich (1986).

Table 2. Number of the orbits among elements for the square plate

		Type of orbit		
Parity of N	Center	Square type I	Square type II	Octagonal
Odd	1	$(N-1)/2$	$(N-1)/2$	$(N-1)(N-3)/8$
Even	0	$N/2$	0	$(N^2-2N)/8$

Table 3. Computing cost for matrix multiplication (17) for each element

		Type of orbit		
μ	Center	Square type I	Square type II	Octagonal
s1, s2, s3, s4	72	80 or 126	76 or 168	110, 138, 168 or 200
d1	152	168 or 300	184 or 400	270, 348, 434 or 528

$$16N^4+127N^3, \tag{24}$$

and for the present method to

$$6N^4+14N^3. \tag{25}$$

Matrix multiplication cost for the present method can be evaluated as below. A set of $N \times N$ elements of the plate is decomposed into four types of orbits among elements. With the use of the number of each orbit listed in Table 2 and the cost for matrix multiplication (17) for each element in Table 3, the cost for the whole plate is evaluated to $232N^2$, which is far smaller in order than the cost (24) or (25) for sweep out. The overall cost by the present method, which is the sum of eqn (25) and $232N^2$, is given by

$$6N^4+14N^3+232N^2, \tag{26}$$

which is approximately 3/8 of the cost (24) for the direct method for large N.

Figure 9 depicts for various numbers of division N the numerical efficiency of the present method relative to the direct method. For the present method the procedures presented in the previous section have all been employed to enhance its numerical efficiency. The solid lines denote the computing time by the present method with and without parallel computation; the dashed one expresses that by the direct method; and the computing time is normalized regarding that of the direct method for $N = 9$. The computing time for $N = 9$ is reduced for the present method more than 20% compared with the direct method. This reduction, however, is smaller than an analytically predicted reduction of 60% by eqns (24) and (26) for $N = 9$. Such smaller reduction is attributable to additional cost for the block-diagonalization analysis in the computer program. Nonetheless, such additional cost will be of the order of (N^2), and is expected to be reduced for large N. With the aid of five

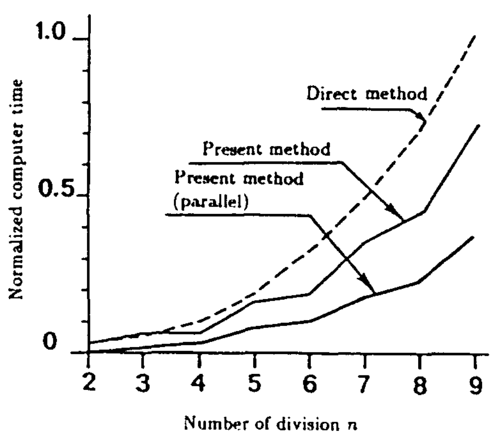

Fig. 9. Numerical efficiency of the present method relative to the direct one.

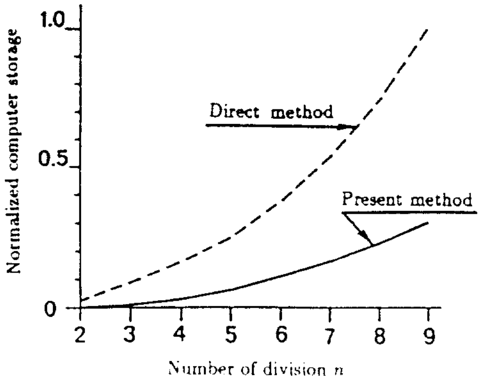

Fig. 10. Requisite computer storage of the present method relative to the direct one.

parallel units the computing time has been further reduced by more than 65%. The time will be further reduced when the computation for each block is performed by parallel machines with many parallel units.

A more important feature of the present method is its lower demand on the computer memory. If the computation for each block is performed in sequence, for block matrices $\tilde{K}^{\mu}$ only the storage for one with the greatest size needs to be reserved. Further, the storage for H^{μ} can be almost nullified by directly computing from formula for H^{μ}. Figure 10 compares the computer memory required for the stiffness matrix for the present and the direct methods, where the direct method demands the storage for the half band matrix. The former, which demands only 30% as much memory as the latter, can save a lot of computer memory.

6. CONCLUSION

In this paper a block-diagonalization analysis of isotropic symmetric plates is presented. The geometric symmetry of the plate has been fully exploited by means of the concept of "augmented" orbit. The present method achieves numerical efficiency and demands less computer storage. Its numerical efficiency has been further upgraded with the use of parallel computation. Although numerical examples are presented only for square plates ($n = 4$), its efficiency will be enhanced as n increases, due to the increase of the number of square blocks, as has been demonstrated by Ikeda and Murota (1991).

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